**EXAMPLE 2.20 (ON TEST)**

**Use the pumping lemma to show that the language *B* = {anbncn | *n ≥* 0} is not context free.**

**We assume that B is a CFL and obtain a contradiction.**

**Let m be the pumping length for *B* that is guaranteed to exist by the pumping lemma.**

**Select the string *s =* ambmcm 🡺 |s| = m+m+m = 3m *≥ m* (Clearly *s* is a member of *B* and of length at least *m).***

**The pumping lemma states that *s* can be pumped, but we show that it cannot.**

**In other words, we show that no matter how we divide s into *uvxyz,* one of the three conditions of the lemma is violated.**

**First, condition 2 stipulates that either *v* or *y* is nonempty. Then we consider one of two cases, depending on whether substrings *v* and *y* contain more than one type of alphabet symbol.**

**1. When both *v* and *y* contain only one type of alphabet symbol, *v* does not contain both a's and b's or both b's and c's, and the same holds for *y.***

**In this case, the string *uv2xy2z* cannot contain equal numbers of a's, b's, and c's. Therefore, it cannot be a member of *B.* That violates condition 1 of the lemma and is thus a contradiction.**

**2. When either *v* or *y* contain more than one type of symbol *uv2xy2z* may contain equal numbers of the three alphabet symbols but won't contain them in the correct order. Hence it cannot be a member of *B* and a con­tradiction occurs.**

**One of these cases must occur. Because both cases result in a contradiction, a contradiction is unavoidable. So, the assumption that *B* is a CFL must be false. Thus, we have proved that *B* is not a CFL.**

**YET ANOTHER PROOF THAT B = {anbncn | n ≥ 0} is not a CFL.**

**Suppose that B is a CFL and thus B = L(G) for some CFG G. Let m be the constant specified by the pumping lemma (PL).**

**Then, w = aNbNcN  with N ≥ m/3 is in L(G) and has the representation w = uvxyz such that v or y is not the empty string Note that |w| ≥ m).**

**Since B is a CFL, uvixyiz ∈L(G) for each i = 0, 1, 2, ...**

**Since |vxy| ≤ m, *vxy* can contain at most 2 of the 3 symbols from {a, b, c}.**

**Since |vy| > 0, v and y together contain at least one symbol.**

**Consider the string uv2xy2z.**

**This string contains additional occurrences of the symbols contained in v and y;**

**Therefore, this string cannot contain equal number of all 3 symbols and thus is not in L(G). Contradiction!!!**

**w not only fails to be an element of B but also fails to be an element of the larger language:**

**L = {w in {a, b, c}\* | w has equal numbers of a’s, b’s and c’s}**

**EXAMPLE 2.21 (ON TEST)**

**Let *C* = {aibjck| 0 < *i < j* < *k}.* We use the pumping lemma to show that *C* is not a CFL.**

**This language is similar to language *B* in Example 2.20, but proving that it is not context free is a bit more complicated.**

**Assume that *C* is a CFL and obtain a contradiction. Let m be the pumping length given by the pumping lemma.**

**We use the string *s* = ambm+1cm+2 that we used earlier, but this time we must "pump down" as well as "pump up." Let *s = uvxyz* and again consider the two cases that occurred in Example 2.20.**

**1. When both *v* and *y* contain only one type of alphabet symbol, *v* does not contain both a's and b's or both b's and c's, and the same holds for *y.* Note that the reasoning used previously in case 1 no longer applies. The reason is that *C* contains strings with unequal numbers of a's, b's, and c's as long as the numbers are not decreasing. We must analyze the situation more carefully to show that *s* cannot be pumped. Observe that because *v* and *y* contain only one type of alphabet symbol, one of the symbols a, b, or c doesn't appear in *v* or *y.* We further subdivide this case into three subcases according to which symbol does not appear.**

**i. The a's do not appear. Then we try pumping down to obtain the string *uv°xy°z = uxz.* That contains the same number of a's as s does, but it contains fewer b's or fewer c's. Therefore, it is not a mem­ber of *C,* and a contradiction occurs.**

**ii. The b's do not appear. Then either a's or c's must appear in *v* or *y* be­cause both can't be the empty string. If a's appear, the string *uv2xy2z* contains more a's than b's, so it is not in *C.* If c's appear, the string *uv°xy°z* contains more b's than c's, so it is not in *C.* Either way a contradiction occurs.**

**iii. The c's do not appear. Then the string *uv2xy2z* contains more a's or more b's than c's, so it is not in C, and a contradiction occurs.**

**2. When either *v* or *y* contain more than one type of symbol, *uv2xy2 z* will not contain the symbols in the correct order. Hence it cannot be a member of *C,* and a contradiction occurs.**

**Thus, we have shown that *s* cannot be pumped in violation of the pumping lemma and that *C* is not context free.**